

雨课堂 Rain Classroom

一、无穷小的比较

例如：当$x \to 0$时，$3x, x^2, \sin x$，都是无穷小.

$\lim\limits_{x\to 0}\dfrac{x^2}{3x}=0$，　　x^2比$3x$要快得多；

$\lim\limits_{x\to 0}\dfrac{\sin x}{3x}=\dfrac{1}{3}$，　　$\sin x$与$3x$大致相同；

极限不同，反映了无穷小趋向于零的“快慢”程度不同.

定义：设α, β是同一过程中的两个无穷小，且$\alpha \neq 0$.

(1)如果$\lim\dfrac{\beta}{\alpha}=0$，就说$\beta$是$\alpha$的高阶无穷小，记作$\beta=o(\alpha)$；

(2)如果$\lim\dfrac{\beta}{\alpha}=C(C\neq 0)$，就说$\beta$与$\alpha$是同阶无穷小；特殊地，

如果$\lim\dfrac{\beta}{\alpha}=1$，则称$\beta$与$\alpha$是等价无穷小，记作$\alpha \sim \beta$；

(3)如果$\lim\dfrac{\beta}{\alpha^k}=C(C\neq 0, k>0)$，就说$\beta$是关于$\alpha$的$k$阶无穷小.

例1　证明：当$x \to 0$时，$4x\tan^3 x$为x的四阶无穷小.

解　$\lim\limits_{x\to 0}\dfrac{4x\tan^3 x}{x^4} = 4\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^3 = 4$，　故结论成立.

例2　当$x \to 0$时，求$\tan x - \sin x$关于x的阶数.

解　$\because \lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3} = \lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\cdot\dfrac{1-\cos x}{x^2}\right)=\dfrac{1}{2}$,

$\therefore \tan x - \sin x$为$x$的三阶无穷小.

$\because \lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$,　　$\therefore$当$x \to 0$时，$\sin x \sim x$;

$\because \lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$,　　$\therefore$当$x \to 0$时，$\tan x \sim x$;

$\because \lim\limits_{x\to 0}\dfrac{\arctan x}{x} \xlongequal{t=\arctan x} \lim\limits_{t\to 0}\dfrac{t}{\tan t}=1$,　$\therefore$当$x \to 0$时，$\arctan x \sim x$;

$\because \lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+x}-1}{x}=\dfrac{1}{n} \Rightarrow \lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+x}-1}{\frac{x}{n}}=1$,　$\therefore$当$x \to 0$时，$\sqrt[n]{1+x}-1 \sim \dfrac{x}{n}$;

$$\because \lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}=1, \qquad \therefore \text{当}x\to 0\text{时}, 1-\cos x\sim\frac{1}{2}x^2;$$

$$\because \lim_{x\to 0}\frac{\tan x-\sin x}{\frac{1}{2}x^3}=1, \qquad \therefore \text{当}x\to 0\text{时}, \tan x-\sin x\sim\frac{1}{2}x^3;$$

$$\because \lim_{x\to 0}\frac{\log_a(1+x)}{\frac{x}{\ln a}}=1, \qquad \therefore \text{当}x\to 0\text{时}, \log_a(1+x)\sim\frac{x}{\ln a};$$

$$\lim_{x\to 0}\frac{\log_a(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\log_a(1+x)=\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}=\log_a e=\frac{\ln e}{\ln a}=\frac{1}{\ln a}$$

$$\because \lim_{x\to 0}\frac{a^x-1}{x\ln a}=1, \qquad \therefore \text{当}x\to 0\text{时}, a^x-1\sim x\ln a;$$

$$\text{令}u=a^x-1\Rightarrow x=\log_a(1+u)\Rightarrow\frac{a^x-1}{x}=\frac{u}{\log_a(1+u)}\Rightarrow\lim_{x\to 0}\frac{a^x-1}{x}=\ln a$$

常用等价无穷小：　当$x \to 0$时，

若$\lim\limits_{x\to 0}\alpha(x)=\lim\limits_{x\to 0}\beta(x)=0$，且$\alpha(x)\sim\beta(x)$，
又若$\lim f(x)=0$，则有$\alpha[f(x)]\sim\beta[f(x)]$.

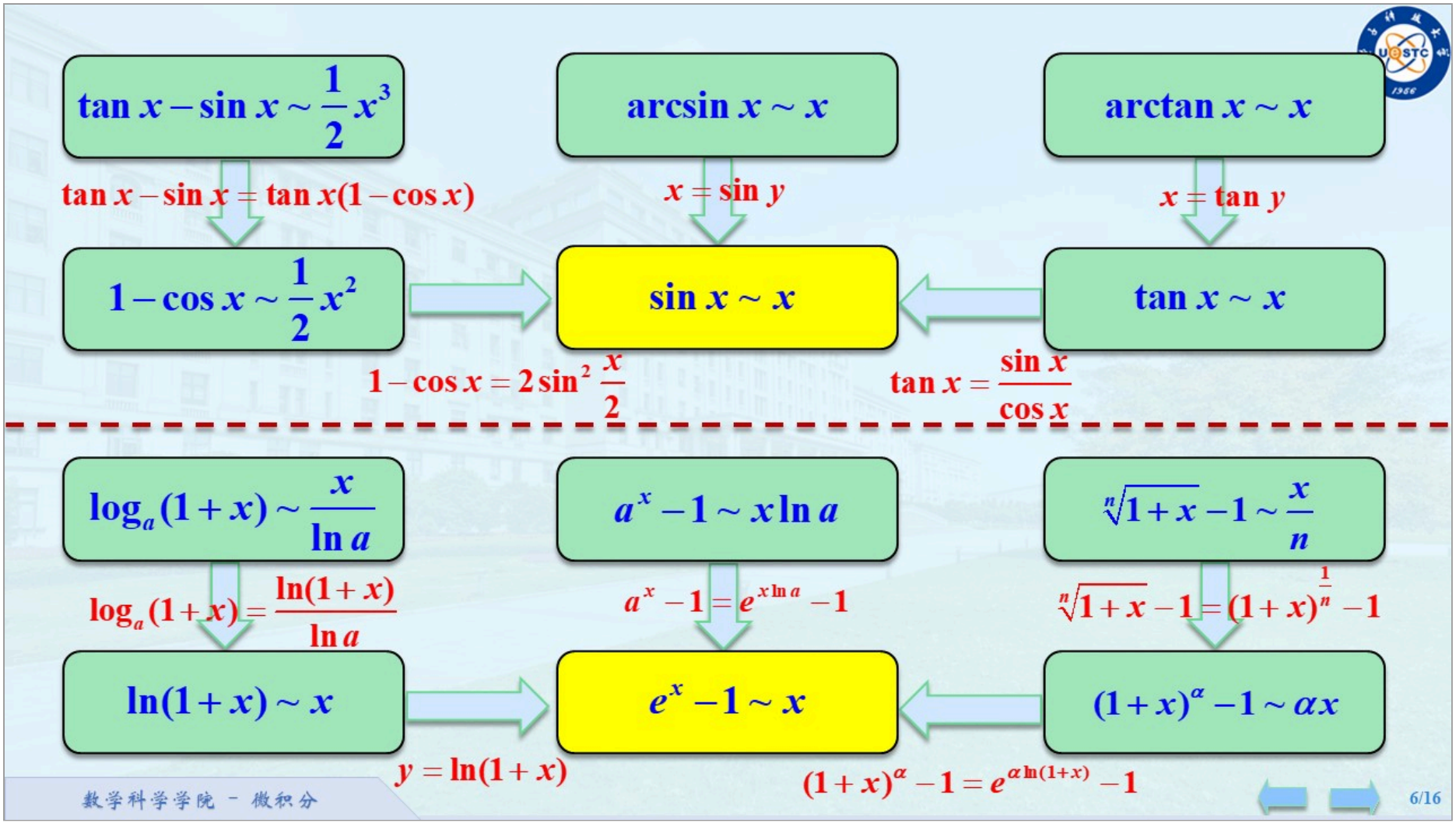
$\sin x\sim x$　　$\arcsin x\sim x$　　$\tan x\sim x$

$\arctan x\sim x$　　$1-\cos x\sim\frac{1}{2}x^2$　　$\tan x-\sin x\sim\frac{1}{2}x^3$

$a^x-1\sim x\ln a$　　$e^x-1\sim x$　　$\log_a(1+x)\sim\frac{x}{\ln a}$

$\ln(1+x)\sim x$　　$\sqrt[n]{1+x}-1\sim\frac{x}{n}$　或　$(1+x)^\alpha-1\sim\alpha x$

以上等价无穷小的关系同学们**必须熟练掌握**，以后经常会使用.


$\tan x-\sin x\sim\frac{1}{2}x^3$
$\arcsin x\sim x$
$\arctan x\sim x$
$\tan x-\sin x=\tan x(1-\cos x)$
$x=\sin y$
$x=\tan y$
$1-\cos x\sim\frac{1}{2}x^2$
$\sin x\sim x$
$\tan x\sim x$
$1-\cos x=2\sin^2\frac{x}{2}$
$\tan x=\frac{\sin x}{\cos x}$
$\log_a(1+x)\sim\frac{x}{\ln a}$
$a^x-1\sim x\ln a$
$\sqrt[n]{1+x}-1\sim\frac{x}{n}$
$\log_a(1+x)=\frac{\ln(1+x)}{\ln a}$
$a^x-1=e^{x\ln a}-1$
$\sqrt[n]{1+x}-1=(1+x)^{\frac{1}{n}}-1$
$\ln(1+x)\sim x$
$e^x-1\sim x$
$(1+x)^\alpha-1\sim\alpha x$
$y=\ln(1+x)$
$(1+x)^\alpha-1=e^{\alpha\ln(1+x)}-1$
数学科学学院 - 微积分
6/16

二、等价无穷小替换

定理（等价无穷小替换定理）

设$\alpha \sim \alpha', \beta \sim \beta' \left(\alpha' \neq 0, \beta' \neq 0\right)$，且$\lim \dfrac{\beta'}{\alpha'} = A$或$\infty$，则

$$\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}.$$

证 $\lim \dfrac{\beta}{\alpha} = \lim \left(\dfrac{\beta}{\beta'} \cdot \dfrac{\beta'}{\alpha'} \cdot \dfrac{\alpha'}{\alpha} \right) = \lim \dfrac{\beta}{\beta'} \cdot \lim \dfrac{\beta'}{\alpha'} \cdot \lim \dfrac{\alpha'}{\alpha} = \lim \dfrac{\beta'}{\alpha'}.$

推论 若$x \to x_0$时$\alpha(x)$与$\beta(x)$是等价无穷小，且

$\lim\limits_{x \to x_0} \left[\beta(x) \cdot f(x)\right] = A$（或$\infty$）. 则

$$\lim_{x \to x_0} \left[\alpha(x) \cdot f(x)\right] = \lim_{x \to x_0} \left[\beta(x) \cdot f(x)\right] = A \left(\text{或} \infty\right).$$

多选题 1分

当$x \to x_0$时,$\alpha(x)$与$\beta(x)$是等价无穷小,下面哪些做法正确?

A $\lim\limits_{x\to x_0} \dfrac{\alpha(x)-f(x)}{h(x)} = \lim\limits_{x\to x_0} \dfrac{\beta(x)-f(x)}{h(x)}$

B $\lim\limits_{x\to x_0} \dfrac{\alpha(x)f(x)}{h(x)} = \lim\limits_{x\to x_0} \dfrac{\beta(x)f(x)}{h(x)}$

C $\lim\limits_{x\to x_0} \dfrac{f(x)}{\alpha(x)-h(x)} = \lim\limits_{x\to x_0} \dfrac{f(x)}{\beta(x)-h(x)}$

D $\lim\limits_{x\to x_0} \dfrac{f(x)-h(x)}{\alpha(x)} = \lim\limits_{x\to x_0} \dfrac{f(x)-h(x)}{\beta(x)}$

雨课堂 Rain Classroom

例3 求 $\lim\limits_{x\to 0}\dfrac{\tan^2 2x}{1-\cos x}$.

解 ∵当$x\to 0$时, $1-\cos x\sim\dfrac{1}{2}x^2$, $\tan 2x\sim 2x$.

$$\therefore \text{原式}=\lim_{x\to 0}\frac{(2x)^2}{\frac{1}{2}x^2}=8.$$

例4 求 $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 2x}$.

解 ∵当$x\to 0$时, $\sin 2x\sim 2x$, $\tan x-\sin x\sim\dfrac{1}{2}x^3$,

$$\therefore \text{原式}=\lim_{x\to 0}\frac{\frac{1}{2}x^3}{(2x)^3}=\frac{1}{16}.$$

例5 求 $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{\sqrt[3]{1+x^2}-1}$

解 $\because x\to 0$时，$1-\cos 2x\sim\dfrac{1}{2}(2x)^2$，$\sqrt[3]{1+x^2}-1\sim\dfrac{x^2}{3}$

$$\therefore \lim_{x\to 0}\frac{1-\cos 2x}{\sqrt[3]{1+x^2}-1}=\lim_{x\to 0}\frac{\frac{1}{2}(2x)^2}{\frac{x^2}{3}}=6$$

例6 求 $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^2(\sqrt{1+x}-1)}$.

解 $\because x\to 0$，$\tan x-\sin x\sim\dfrac{1}{2}x^3$，$\sqrt{1+x}-1\sim\dfrac{1}{2}x$

$$\therefore \lim_{x\to 0}\frac{\tan x-\sin x}{x^2(\sqrt{1+x}-1)}=\lim_{x\to 0}\frac{\frac{1}{2}x^3}{x^2\cdot\frac{1}{2}x}=1.$$