例1 求 $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

解 原式 $=\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}}{x^2}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\sin^2\dfrac{x}{2}}{\left(\dfrac{x}{2}\right)^2}=\dfrac{1}{2}\lim\limits_{x\to 0}\left(\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2=\dfrac{1}{2}.$

例2 求 $\lim\limits_{x\to 0}\dfrac{x^2\sin\dfrac{1}{x}}{\sin x}$.

解 原式 $=\lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot x\sin\dfrac{1}{x}=\lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=0.$

例3 求 $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

解 原式 $=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos x}=1.$

例4　求 $\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}$　$(m,n\neq 0)$.

解　$\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}=\lim\limits_{x\to 0}\dfrac{\sin mx}{mx}\cdot\dfrac{mx}{nx}\cdot\dfrac{nx}{\sin nx}$

$=\lim\limits_{x\to 0}\dfrac{\sin mx}{mx}\cdot\lim\limits_{x\to 0}\dfrac{mx}{nx}\cdot\lim\limits_{x\to 0}\dfrac{nx}{\sin nx}=1\cdot\dfrac{m}{n}\cdot 1=\dfrac{m}{n}.$

例5　求 $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.

解　$\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\cdot\dfrac{1-\cos x}{x^2}\right)$

$=\lim\limits_{x\to 0}\dfrac{\tan x}{x}\cdot\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=1\cdot\dfrac{1}{2}=\dfrac{1}{2}.$

例6　求 $\lim\limits_{x\to\pi}\dfrac{\sin 3x}{\tan 5x}$.　解　令 $t=\pi-x\Rightarrow x=\pi-t$ ⇒当 $x\to\pi$ 时, $t\to 0$, 则

$$\text{原式}=\lim_{t\to 0}\frac{\sin(3\pi-3t)}{\tan(5\pi-5t)}=-\lim_{t\to 0}\frac{\sin 3t}{\tan 5t}=-\lim_{t\to 0}\frac{\dfrac{\sin 3t}{3t}}{\dfrac{\tan 5t}{5t}}\cdot\frac{3}{5}=-\frac{3}{5}.$$

例7　求 $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi}{2}x$. 解：令 $u=1-x\Rightarrow x=1-u$, 且 $x\to 1\Rightarrow u\to 0$, 故

$$\text{原式}=\lim_{u\to 0}u\tan\frac{\pi}{2}(1-u)=\lim_{u\to 0}u\cot\frac{\pi}{2}u$$

$$=\lim_{u\to 0}\frac{u}{\tan\dfrac{\pi}{2}u}=\frac{2}{\pi}\cdot\lim_{u\to 0}\frac{\dfrac{\pi}{2}u}{\tan\dfrac{\pi}{2}u}=\frac{2}{\pi}.$$

结论：若 $\lim f(x)=0$, 则

$$\lim\frac{\sin f(x)}{f(x)}=1$$

$$(2)\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \qquad 1^{\infty}$$

银行说："大家一起来发财吧!",它给的年利率为100%,结算时间你来定.

如果你有1块钱,一年结算一次,则一年后得到：$1\cdot(1+1)=(1+1)^{1}=2;$

半年结算一次,则一年后得到：$(1+\frac{1}{2})(1+\frac{1}{2})=(1+\frac{1}{2})^{2}=2.25$

4个月结算一次,则一年后得到：$(1+\frac{1}{3})(1+\frac{1}{3})(1+\frac{1}{3})=(1+\frac{1}{3})^{3}=2.3704$

1个月结算一次,则一年后得到：$(1+\frac{1}{12})^{12}=2.6130$

1天结算一次,则一年后得到：$(1+\frac{1}{365})^{365}=2.7146$

1小时结算一次,则一年后得到：$(1+\frac{1}{8760})^{8760}=2.7181$



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首先证明：$\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$，设 $x_n=\left(1+\frac{1}{n}\right)^n$，则

$$(a+b)^n=\sum_{i=0}^{n}C_n^i a^i b^{n-i}$$

$$x_n=1+\frac{n}{1!}\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\cdots+\frac{n(n-1)\cdots(n-n+1)}{n!}\cdot\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right).$$

类似的，$x_{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$

$$+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

显然 $x_{n+1}>x_n$，　$\therefore\{x_n\}$单调递增；

$$x_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

$$<1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}<1+1+\frac{1}{2}+\cdots+\frac{1}{2^{n-1}}$$

$$=1+\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}=3-\left(\frac{1}{2}\right)^{n-1}<3,$$

$\therefore \{x_n\}$是有界的; $\therefore \lim\limits_{n\to\infty} x_n$ 存在.

记 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ $(e = 2.71828\cdots)$

再证：$\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$，当$x>1$时，$\exists$正整数$n$，使得$n\le x\le n+1$，

$$\left(1+\frac{1}{n+1}\right)^{n}\le\left(1+\frac{1}{x}\right)^{x}\le\left(1+\frac{1}{n}\right)^{n+1}$$，且$x\to+\infty$时，$n\to\infty$

$$\text{而}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=e,$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{-1}=e,$$

由夹逼定理得：$\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e.$

对 $\lim\limits_{x\to-\infty}\left(1+\dfrac{1}{x}\right)^{x}$，令 $t=-x$，

$$\therefore \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{t\to+\infty}\left(1-\frac{1}{t}\right)^{-t}=\lim_{t\to+\infty}\left(\frac{t-1}{t-1+1}\right)^{-t}=\lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t}$$

$$=\lim_{t\to+\infty}\left(1+\frac{1}{t-1}\right)^{t-1}\left(1+\frac{1}{t-1}\right)=e.$$

$$\therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

令 $x=\dfrac{1}{t}$，$\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=\lim\limits_{t\to\infty}\left(1+\dfrac{1}{t}\right)^{t}=e.$

结论：若 $\lim f(x)=0$，则 $\lim\left[1+f(x)\right]^{\frac{1}{f(x)}}=e$

$\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=e$　　$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

例8 求 $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{x}$ 1^{∞}

解 原式 $=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{1}{-x}\right)^{-x}\right]^{-1}=e^{-1}=\dfrac{1}{e}.$

$a^{bc}=(a^{b})^{c}$

命题：若 $\lim g(x)=A,\lim f(x)=B>0,$
则 $\lim f(x)^{g(x)}=B^{A}=[\lim f(x)]^{\lim g(x)}.$

证 $\lim f(x)^{g(x)}=\lim e^{g(x)\ln f(x)}$

$=e^{\lim g(x)\ln f(x)}=e^{\lim g(x)\cdot\lim \ln f(x)}=e^{A\cdot\ln\lim f(x)}$

$=e^{A\ln B}=e^{\ln B^{A}}=B^{A}=[\lim f(x)]^{\lim g(x)}.$

例9 (1)求 $\lim\limits_{x\to\infty}\left(\dfrac{3+x}{2+x}\right)^{2x}$ 1^{∞}

解 原式 $=\lim\limits_{x\to\infty}\left[\left(1+\dfrac{1}{x+2}\right)^{x+2}\right]^{\frac{2x}{x+2}}=e^{\lim\limits_{x\to\infty}\frac{2x}{x+2}}=e^{2}$

(2) 求 $\lim\limits_{x\to0}\left(1+3\tan^{2}x\right)^{\cot^{2}x}$ $.1^{\infty}$

解 $\lim\limits_{x\to0}\left(1+3\tan^{2}x\right)^{\cot^{2}x}=\lim\limits_{x\to0}\left[\left(1+3\tan^{2}x\right)^{\frac{1}{3\tan^{2}x}}\right]^{3}=e^{3}.$

单选题 1分

极限 $\lim\limits_{n\to\infty}\left(1-\dfrac{m^2}{n^2}\right)^n=(\quad)$

A e^{m^2}　　B 1　　C $e^{\frac{m^2}{n}}$　　D $e^{-\frac{m^2}{n}}$

数学科学学院 - 微积分

21/22

例10 求 $\lim\limits_{n\to\infty}\left(1-\dfrac{m^2}{n^2}\right)^n$

解 $\lim\limits_{n\to\infty}\left(1-\dfrac{m^2}{n^2}\right)^n=\lim\limits_{n\to\infty}\left[\left(1+\dfrac{-m^2}{n^2}\right)^{\frac{n^2}{m^2}}\right]^{\left(\frac{m^2}{n^2}\right)n}=e^0=1.$

例11 求 $\lim\limits_{n\to\infty}\left(1+\dfrac{2nx+x^2}{2n^2}\right)^{-n}$.

解 $\lim\limits_{n\to\infty}\left(1+\dfrac{2nx+x^2}{2n^2}\right)^{-n}=\lim\limits_{n\to\infty}\left[\left(1+\dfrac{2nx+x^2}{2n^2}\right)^{\frac{2n^2}{2nx+x^2}}\right]^{\frac{2nx+x^2}{2n^2}\cdot(-n)}=e^{-x}$

作 业

习题册

- 第8页 ☛ 1
- 第9页 ☛ 5, 6, 7