
电子科技大学
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第五节 极限存在准则
两个重要极限
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一、极限存在准则

1. 夹逼准则

以$x \to x_0$为例

从某时刻后
不等式总成立

准则Ⅰ 如果当$x \in \mathring{U}(x_0, \delta)$时,有

$$(1)\ g(x) \le f(x) \le h(x);\quad (2)\ \lim_{x \to x_0} g(x) = A,\ \lim_{x \to x_0} h(x) = A.$$

那么$\lim\limits_{x \to x_0} f(x)$存在,且等于$A$, 即$\lim\limits_{x \to x_0} f(x) = A$.

证 $\forall \varepsilon > 0$, $\lim\limits_{x \to x_0} g(x) = A \Rightarrow \exists \delta_1 > 0$,当$0 < |x - x_0| < \delta_1$时,有$|g(x) - A| < \varepsilon$

$$\Rightarrow A - \varepsilon < g(x) < A + \varepsilon$$

$\lim\limits_{x \to x_0} h(x) = A \Rightarrow \exists \delta_2 > 0$,当$0 < |x - x_0| < \delta_2$时,有$|h(x) - A| < \varepsilon$

$$\Rightarrow A - \varepsilon < h(x) < A + \varepsilon,$$

$\because$当$0 < |x - x_0| < \delta$时,$g(x) < f(x) < h(x)$, 取$\delta_0 = \min\{\delta_1, \delta_2, \delta\}$, 当$0 < |x - x_0| < \delta_0$时,

有$A - \varepsilon < g(x) < f(x) < h(x) < A + \varepsilon \Rightarrow |f(x) - A| < \varepsilon \Rightarrow \lim\limits_{x \to x_0} f(x) = A$

例1 求 $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right)$.

解 $\because \dfrac{n}{\sqrt{n^2+n}}<\dfrac{1}{\sqrt{n^2+1}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}<\dfrac{n}{\sqrt{n^2+1}}$,

又 $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+n}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\dfrac{1}{n}}}=1$, $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+1}}=\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\dfrac{1}{n^2}}}=1$,

由夹逼定理得 $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right)=1$.

例2 求 $\lim\limits_{x\to 0}\cos x=1$.

解 $\because 0\le 1-\cos x=2\sin^2\dfrac{x}{2}<2\left(\dfrac{x}{2}\right)^2=\dfrac{x^2}{2}$, 且 $\lim\limits_{x\to 0}\dfrac{x^2}{2}=0$

$\therefore \lim\limits_{x\to 0}(1-\cos x)=0\Rightarrow\lim\limits_{x\to 0}\cos x=1$

例3 设$a_i>0(i=1,2,\cdots,m)$,求$\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}$.

解 记$\max\{a_1,a_2,\cdots,a_m\}=a$,则有

$$a=\sqrt[n]{a^n}<\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}\le\sqrt[n]{ma^n}=a\sqrt[n]{m}$$

而$\lim\limits_{n\to\infty}\sqrt[n]{m}=\lim\limits_{n\to\infty}m^{\frac{1}{n}}=\lim\limits_{n\to\infty}e^{\frac{1}{n}\ln m}\xlongequal{u=\frac{1}{n}\ln m}\lim\limits_{u\to0}e^u=1$

由夹逼准则得,$\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}=a=\max\{a_1,a_2,\cdots,a_m\}$.

$a^b=e^{\ln a^b}=e^{b\ln a}\ (a>0)$

记住该结论

2. 单调有界准则

如果数列 $\{x_n\}$满足条件

$x_1 \le x_2 \cdots \le x_n \le x_{n+1} \le \cdots$，单调增加

$x_1 \ge x_2 \cdots \ge x_n \ge x_{n+1} \ge \cdots$，单调减少

单调数列

有上界,即

$\exists M, \forall n$,有$x_n \le M$

有下界,即

$\exists m, \forall n$,有$x_n \ge m$

准则II 单调有界数列必有极限.

本结论也适用于$x \to x_0^+, x \to x_0^-, x \to +\infty$和$x \to -\infty$等四类函数极限.

例4. 设$x_1>0$,且$x_{n+1}=\frac{1}{2}\left(x_n+\frac{a}{x_n}\right)(n=1,2,3,\cdots,a>0)$,证明$\lim\limits_{n\to\infty}x_n$存在并求此极限值.

证　由已知$x_1>0$及递推公式知$x_n>0$,

$$x_n=\frac{1}{2}\left(x_{n-1}+\frac{a}{x_{n-1}}\right)\geq\sqrt{x_{n-1}\cdot\frac{a}{x_{n-1}}}=\sqrt{a}\Rightarrow\{x_n\}\text{有下界}\sqrt{a},(n\geq 2)$$

$$x_{n+1}-x_n=\frac{a}{2x_n}-\frac{x_n}{2}=\frac{a-x_n^2}{2x_n}\leq 0\Rightarrow x_{n+1}\leq x_n\Rightarrow\{x_n\}\text{单调减少},(n\geq 2).$$

根据单调有界准则,$\lim\limits_{n\to\infty}x_n$存在. 令$\lim\limits_{n\to\infty}x_n=A$,得到

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}\frac{1}{2}\left(x_{n-1}+\frac{a}{x_{n-1}}\right)\Rightarrow A=\frac{1}{2}\left(A+\frac{a}{A}\right)\Rightarrow A=\pm\sqrt{a}$$

因$x_n>0$, 故$\lim\limits_{n\to\infty}x_n=A\geq 0$. 所以$\lim\limits_{n\to\infty}x_n=\sqrt{a}$.

能否结合极限的定义完成该题?

证2

假设$\lim\limits_{n\to\infty}x_n=A$,则$\lim\limits_{n\to\infty}x_n=\lim\limits_{n\to\infty}\frac{1}{2}\left(x_{n-1}+\frac{a}{x_{n-1}}\right)\Rightarrow A=\frac{1}{2}\left(A+\frac{a}{A}\right)\Rightarrow A=\pm\sqrt{a}$

因$x_n>0$, 故$\lim\limits_{n\to\infty}x_n\geq 0$. 所以若$\lim\limits_{n\to\infty}x_n$存在,则$\lim\limits_{n\to\infty}x_n=\sqrt{a}$. 下证$\sqrt{a}$是$\{x_n\}$的极限.

$$\left|x_n-\sqrt{a}\right|=\left|\frac{1}{2}\left(x_{n-1}+\frac{a}{x_{n-1}}\right)-\frac{1}{2}\left(\sqrt{a}+\frac{a}{\sqrt{a}}\right)\right|=\frac{1}{2}\left|x_{n-1}-\sqrt{a}+\frac{a}{x_{n-1}}-\sqrt{a}\right|$$

$$=\frac{1}{2}\left|x_{n-1}-\sqrt{a}+\sqrt{a}\,\frac{\sqrt{a}-x_{n-1}}{x_{n-1}}\right|=\frac{1}{2}\left|(x_{n-1}-\sqrt{a})(1-\frac{\sqrt{a}}{x_{n-1}})\right|=\frac{1}{2}\left|1-\frac{\sqrt{a}}{x_{n-1}}\right|\left|x_{n-1}-\sqrt{a}\right|$$

因于$x_n=\frac{1}{2}\left(x_{n-1}+\frac{a}{x_{n-1}}\right)\geq\sqrt{x_{n-1}\cdot\frac{a}{x_{n-1}}}=\sqrt{a}\ (n\geq 2)\Rightarrow\left|1-\frac{\sqrt{a}}{x_{n-1}}\right|<1$

从而$\left|x_n-\sqrt{a}\right|<\frac{1}{2}\left|x_{n-1}-\sqrt{a}\right|<\frac{1}{2^2}\left|x_{n-2}-\sqrt{a}\right|<\cdots<\frac{1}{2^{n-2}}\left|x_2-\sqrt{a}\right|\to 0$

所以$\{x_n\}$的极限存在,且$\lim\limits_{n\to\infty}x_n=\sqrt{a}$.

例5　证明数列: $x_1=\sqrt{a}, x_2=\sqrt{a+\sqrt{a}}, x_3=\sqrt{a+\sqrt{a+\sqrt{a}}}, \cdots (a>0)$的极限 $\lim\limits_{n\to\infty} x_n$存在,并求此极限.　　注意到: $x_n=\sqrt{a+x_{n-1}}$

解　1. $x_1=\sqrt{a}<\sqrt{a+\sqrt{a}}=x_2$, 假设$x_{n-1}<x_n$,由已知得

$$x_n=\sqrt{a+x_{n-1}}<\sqrt{a+x_n}=x_{n+1}\Rightarrow x_n<x_{n+1}\Rightarrow \{x_n\}\text{单调增加.}$$

$$2.\ x_n=\sqrt{a+x_{n-1}}<\sqrt{a+x_n}\Rightarrow x_n^2<a+x_n\Rightarrow x_n<\frac{1}{2}+\frac{\sqrt{1+4a}}{2}\Rightarrow \{x_n\}\text{有上界.}$$

由单调有界准则知$\lim\limits_{n\to\infty} x_n$存在. 令$\lim\limits_{n\to\infty} x_n=A$,

$$\text{由}x_n=\sqrt{a+x_{n-1}}\Rightarrow A=\sqrt{a+A}\Rightarrow A^2-A-a=0\Rightarrow A=\frac{1\pm\sqrt{1+4a}}{2},$$

$$\text{由}x_n>0\Rightarrow \lim_{n\to\infty} x_n=A\geq 0,\ \text{所以}\ \lim_{n\to\infty} x_n=A=\frac{1+\sqrt{1+4a}}{2}.$$

$$2=\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}$$

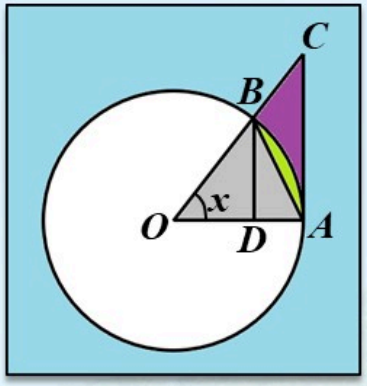
$$3=\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}}$$

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}$$

连分式

二、两个重要极限

(1) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ 记住此公式



设单位圆O,圆心角$\angle AOB=x(0<x<\dfrac{\pi}{2})$,过$A$点作切线,得$\Delta ACO$.扇形$OAB$的圆心角为$x$,$\Delta OAB$高为$BD$,于是$S_{\Delta OAB}=\dfrac{1}{2}\sin x$, $S_{扇形OAB}=\dfrac{1}{2}x$, $S_{\Delta OAC}=\dfrac{1}{2}\tan x$.

$\Rightarrow \sin x<x<\tan x \Rightarrow 1<\dfrac{x}{\sin x}<\dfrac{1}{\cos x}\Rightarrow \cos x<\dfrac{\sin x}{x}<1,$

当$-\dfrac{\pi}{2}<x<0$时上式也成立 $\Rightarrow 0<|x|<\dfrac{\pi}{2}$时,$\cos x<\dfrac{\sin x}{x}<1$

由 $\lim\limits_{x\to 0}\cos x=1$, $\lim\limits_{x\to 0}1=1\Rightarrow \lim\limits_{x\to 0}\dfrac{\sin x}{x}=1.$

单选题 1分

若$\exists \delta > 0$,当$0 < |x - x_0| < \delta$时,$\varphi(x) \neq 0$,

且$\lim\limits_{x \to x_0} \varphi(x) = 0$,则$\lim\limits_{x \to x_0} \dfrac{\sin \varphi(x)}{\varphi^2(x)} = (\quad)$

A 不能确定

B 0

C 1

D ∞

若 $\lim\limits_{x\to x_0}\varphi(x)=0$，则 $\lim\limits_{x\to x_0}\dfrac{\sin\varphi(x)}{\varphi^2(x)}=(\quad)$

解 $\lim\limits_{x\to x_0}\dfrac{\sin\varphi(x)}{\varphi^2(x)}=\lim\limits_{x\to x_0}\left[\dfrac{\sin\varphi(x)}{\varphi(x)}\cdot\dfrac{1}{\varphi(x)}\right]$

由于 $\lim\limits_{x\to x_0}\dfrac{\sin\varphi(x)}{\varphi(x)}\xlongequal{u=\varphi(x)}\lim\limits_{u\to 0}\dfrac{\sin u}{u}=1,$

$$\lim_{x\to x_0}\frac{1}{\varphi(x)}=\infty$$

所以 $\lim\limits_{x\to x_0}\dfrac{\sin\varphi(x)}{\varphi^2(x)}=\infty.$

问题：设 $\lim\limits_{x\to x_0}f(x)=A>0,\lim\limits_{x\to x_0}g(x)=\infty,$

则 $\lim\limits_{x\to x_0}[f(x)\cdot g(x)]=\infty.$

证 $\forall M>0$，对

$\lim\limits_{x\to x_0}f(x)=A>0\Rightarrow$ 对 $\varepsilon=\dfrac{A}{2},\exists\delta_1>0,$

当 $0<|x-x_0|<\delta_1$ 时，有 $|f(x)-A|<\dfrac{A}{2}$

$$\Rightarrow\frac{A}{2}<f(x)<\frac{3}{2}A.$$

$\lim\limits_{x\to x_0}g(x)=\infty\Rightarrow\exists\delta_2>0,$ 当 $0<|x-x_0|<\delta_2$ 时，

有 $|g(x)|>M.$

取 $\delta=\min\{\delta_1,\delta_2\}$，当 $0<|x-x_0|<\delta$ 时，

有 $|f(x)g(x)|>\dfrac{A}{2}M$

$\Rightarrow\lim\limits_{x\to x_0}[f(x)\cdot g(x)]=\infty.$

例1 求 $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$.

解 原式 $=\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}}{x^2}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\sin^2\dfrac{x}{2}}{\left(\dfrac{x}{2}\right)^2}=\dfrac{1}{2}\lim\limits_{x\to 0}\left(\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2=\dfrac{1}{2}.$

例2 求 $\lim\limits_{x\to 0}\dfrac{x^2\sin\dfrac{1}{x}}{\sin x}$.

解 原式 $=\lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot x\sin\dfrac{1}{x}=\lim\limits_{x\to 0}\dfrac{x}{\sin x}\cdot\lim\limits_{x\to 0}x\sin\dfrac{1}{x}=0.$

例3 求 $\lim\limits_{x\to 0}\dfrac{\tan x}{x}$.

解 原式 $=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{1}{\cos x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}\cdot\lim\limits_{x\to 0}\dfrac{1}{\cos x}=1.$